SPECTRAL CONDITIONS FOR LOCAL NONDETERMINISM

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Let $X(t)$ be a real Gaussian process with stationary increments and spectral distribution function $F(x)$. Put $\phi(t) = F(\infty) - F(1/t)$. Sufficient conditions in terms of F are given for the process to be locally ϕ-nondeterministic. These are formulated for discrete and absolutely continuous functions F. The results in the discrete case are applied to the analysis of the local time of a random Fourier series with i.i.d. coefficients. The class of distributions of the coefficients includes not only the normal distribution but others such as the symmetric stable distribution.

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1. Introduction and summary

Let $X(t)$, $t \geq 0$, be a separable Gaussian process with mean 0, and let J be an open interval on the t-axis. Assume that there exists $d > 0$ such that

$$\begin{aligned} &EX^2(t) > 0, \quad t \in J, \quad \text{and} \\ &E(X(t) - X(s))^2 > 0 \quad \text{for} \quad 0 < |t-s| < d, \quad s, t \in J. \end{aligned} \tag{1.1}$$

The concept of local nondeterminism (LND) was introduced by the author in [2]. According to Lemmas 2.1 and 2.2 of that paper, the definition of LND is equivalent to the following: For every $m \geq 2$, let $t_1 < t_2 < \cdots < t_m$ be variable ordered points in J; then the determinant of the covariance matrix of the m standardized random variables,

$$\frac{X(t_1)}{(\operatorname{Var} X(t_1))^{1/2}}, \frac{X(t_2) - X(t_1)}{(\operatorname{Var}(X(t_2) - X(t_1)))^{1/2}}, \ldots, \frac{X(t_m) - X(t_{m-1})}{(\operatorname{Var}(X(t_m) - X(t_{m-1})))^{1/2}}$$

is, as a function of $t_1 < \cdots < t_m$, bounded away from 0. The concept was extended by Cuzick [4], who defined local ϕ-nondeterminism, LND (ϕ), by replacing the

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variances in the denominators above by the values of a function $\phi(t_j - t_{j-1})$, where ϕ is a positive function. As an immediate consequence of [2], Lemma 2.3, $X(t)$ has LND(ϕ) if and only if

$$\lim_{c \downarrow 0} \liminf_{t_m - t_1 \leq c} \operatorname{Var}\left\{ \frac{b_1 X(t_1)}{\phi^{1/2}(t_1)} + \sum_{j=2}^{m} b_j \frac{X(t_j) - X(t_{j-1})}{\phi^{1/2}(t_j - t_{j-1})} \right\} > 0, \tag{1.2}$$

for every nonzero vector $(b_1, \ldots, b_m)$, for $m \geq 2$.

Suppose that $X(t)$ also has stationary increments. Assume $X(0) = 0$ a.s., and define the incremental variance function $\sigma^2(t) = EX^2(t) = E(X(s+t) - X(s))^2$. The latter has the well known representation

$$\sigma^2(t) = \tfrac{1}{4} \int_{-\infty}^{\infty} |e^{ixt} - 1|^2 (1 + x^2) x^{-2} \, dF(x), \tag{1.3}$$

where F is the spectral distribution function. Since X is real, F is symmetric, and the representation becomes

$$\sigma^2(t) = \int_0^{\infty} (1 - \cos xt)(1 + x^2) x^{-2} \, dF(x),$$

so that $F(x)$, for $x \geq 0$ determines σ^2. In this paper we consider processes having one of two types of spectral functions. The first type is the absolutely continuous F with the density function $f(x)$:

$$F(x) = \int_0^x f(y) \, dy. \tag{1.4}$$

The second is the discrete type:

$$F(x) = \sum_{0 \leq n \leq x} f_n, \tag{1.5}$$

where $f_n \geq 0$, and $\sum f_n < \infty$. Our main result is:

Theorem 1.1. *Let $X(t)$, $t \geq 0$, be a Gaussian process with mean 0 and stationary increments, and with spectral function of the form* (1.4) *or* (1.5). *Define*

$$\phi(t) = F(\infty) - F(1/t), \quad t > 0. \tag{1.6}$$

Then there exists $d > 0$ sufficiently small such that LND(ϕ) *holds for $X(t)$, $0 < t < d$, if (two cases)*

$$0 < \liminf_{x \to \infty} \frac{xf(x)}{F(\infty) - F(x)}, \qquad \limsup_{x \to \infty} \frac{xf(x)}{F(\infty) - F(x)} < 2, \tag{1.7}$$

under (1.4); *and*

$$0 < \liminf_{n \to \infty} \frac{nf_n}{\sum_{j>n} f_j}, \qquad \limsup_{n \to \infty} \frac{nf_n}{\sum_{j>n} f_j} < 2, \tag{1.8}$$

under (1.5).

The proof is given in Section 3. Applications to Gaussian and non-Gaussian random Fourier series are presented in Section 4.

Sufficient conditions on $\sigma^2(t)$ and on $F(x)$ for LND(σ^2), ordinary local nondeterminism as defined after (1.1), were given by the author in a series of several papers, and summarized in [2]. Most of the conditions on $\sigma^2(t)$ involved the regular variation of $\sigma^2(t)$ of index α, $0<\alpha<2$, for $t\to 0$. The conditions on F were not as neat, and required that $F(x)$ have an absolutely continuous component with a regularly varying density for large x. The conditions of Cuzick [4] for LND(ϕ) also required the regular variation of ϕ. Recent work in this area has also been done by Miroshin [6].

By the uniqueness of the representation (1.3), the function $\sigma^2(t)$ and the function $\phi(t)$ in (1.6) uniquely determine each other. Furthermore, if either is regularly varying, then, by the general form of Karamata's theorem (see [7]) so is the other, and the two are, up to a constant multiple, asymptotically equal for $t\to 0$. Thus, in that case, LND(σ^2) and LND(ϕ) are equivalent. However, in general, in the absence of regular variation, it is difficult to formulate nice conditions on F for LND(σ^2) because the asymptotic relations for σ^2 and F are not mutually deducible. The main contribution of the present work is showing that if ϕ is chosen as the particular function in (1.7), then conditions for LND(ϕ) can be stated very simply in terms of F without the requirement of regular variation. Furthermore, these conditions are stated for both the discrete and absolutely continous cases.

The conditions (1.7) and (1.8) are more general than regular variation. Let us demonstrate this in the case of (1.7); the other case is similar. Since

$$\mathrm{d}[\log(F(\infty)-F(x))]=-f(x)/F(\infty)-F(x)),$$

we have the identity

$$\frac{F(\infty)-F(x)}{F(\infty)-F(y)}=\exp\left\{\int_x^y \frac{zf(z)}{F(\infty)-F(z)}\,\mathrm{d}z/z\right\}, \tag{1.9}$$

for any x and y. If

$$\alpha=\lim_{z\to\infty}\frac{zf(z)}{F(\infty)-F(z)}, \tag{1.10}$$

exists, then $F(\infty)-F(x)$ is of regular variation of index $-\alpha$ for $x\to\infty$. Conversely, by a generalization of a theorem of Karamata, due to deHaan [5, p.15], (1.10) is necessary for $f(x)$ to be of regular variation of index $-\alpha-1$. As an immediate result, we note that the conditions of Theorem 1.1 hold if (f_n) is a regularly varying sequence of index $-\alpha-1$ for $n\to\infty$, or if $f(x)$ is a regularly varying function of the same index for $x\to\infty$.

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2. Preliminary results

In this section we obtain several asymptotic properties of the function ϕ defined by (1.6) under the assumptions (1.7) or (1.8).

Lemma 2.1. *Under* (1.7) *or* (1.8):

$$\lim_{t\to 0} \phi(t)/t^2=\infty, \tag{2.1}$$

$$\limsup_{s,t\to 0, s/t\leq c} \phi(s)/\phi(t)<\infty \quad \textit{for} \quad c<\infty, \tag{2.2}$$

$$\liminf_{s,t\to 0, s/t\geq c} \phi(s)/\phi(t)>0 \quad \textit{for} \quad c>0, \tag{2.3}$$

$$\lim_{s,t\to 0, s/t\to 0} \frac{\phi(t)/t^2}{\phi(s)/s^2}=0. \tag{2.4}$$

Proof. Under (1.7) in the case (1.4), there exists ε, $0<\varepsilon<1$, such that

$$\varepsilon<\frac{zf(z)}{F(\infty)-F(z)}<2-\varepsilon, \tag{2.5}$$

for all sufficiently large z. Therefore, (1.9) implies

$$(s/t)^{2-\varepsilon}\leq \phi(s)/\phi(t)\leq (s/t)^{\varepsilon}, \tag{2.6}$$

for $s<t$, for sufficiently small $t>0$. The assertions of the lemma follow from (2.6).

Under (1.8) in the case (1.5), the analogue of (2.5) is

$$\varepsilon<\frac{nf_n}{\sum_{j>n} f_j}<2-\varepsilon \tag{2.7}$$

for all sufficiently large n. Put $q_n=\sum_{j>n} f_j$, so that $f_n=q_{n-1}-q_n$, and (2.7) becomes

$$\varepsilon q_n<n(q_{n-1}-q_n)<(2-\varepsilon)q_n,$$

which is equivalent to

$$1+\varepsilon/n<\frac{q_{n-1}}{q_n}<1+(2-\varepsilon)/n.$$

By iterating, we obtain, for $m<n$,

$$\prod_{j=m+1}^{n}(1+\varepsilon/j)<q_m/q_n<\prod_{j=m+1}^{n}\left(1+\frac{2-\varepsilon}{j}\right). \tag{2.8}$$

The right hand member above is at most equal to

$$\exp\left((2-\varepsilon)\sum_{j=m+1}^{n} 1/j\right)\leq \exp\left((2-\varepsilon)\log\frac{n+1}{m}\right)=\left(\frac{n+1}{m}\right)^{2-\varepsilon}.$$

Since $e^{x/2} \leq 1+x$ for all sufficiently small $x>0$, it also follows that the left hand member of (2.8) is at least equal to

$$\exp\left(\frac{\varepsilon}{2}\sum_{j=m+1}^{n-1} 1/j\right) > \left(\frac{n-1}{m+1}\right)^{\varepsilon/2}$$

It then follows from (2.8) that

$$\left(\frac{n-1}{m+1}\right)^{\varepsilon/2} < \frac{q_m}{q_n} < \left(\frac{n+1}{m}\right)^{2-\varepsilon} \tag{2.9}$$

for all $m<n$ and sufficiently large m. Since, by the definition of q_n and that of ϕ in (1.6), we have $\phi(t)=q_{[1/t]}$, the statements in the lemma follow from (2.9) in the case (1.5) in the same way as they follow from (2.6) in the case (1.4).

Lemma. 2.2. *Under* (1.7), *there exists* $\varepsilon>0$ *such that for all sufficiently large* b *and* $c>b$, *and every nonnegative measurable function* $g(x)$,

$$\int_b^c g(x)f(x)\,dx \geq \varepsilon\phi(1/c)\int_b^c x^{-1}g(x)\,dx. \tag{2.10}$$

Proof. By (2.5),

$$\int_b^c g(x)f(x)\,dx \geq \varepsilon \int_b^c g(x)x^{-1}\phi(1/x)\,dx$$
$$\geq \varepsilon\phi(1/c)\int_b^c g(x)x^{-1}\,dx.$$

Similarly, we have:

Lemma 2.3. *Under* (1.8), *there exists* $\varepsilon>0$ *such that for all sufficiently large* $b>0$ *and* $c>b$, *and every nonegative sequence* (g_n),

$$\sum_{b\leq j\leq c} g_jf_j \geq \varepsilon\phi(1/c)\sum_{b\leq j\leq c} g_j/j. \tag{2.11}$$

3. Proof of Theorem 1.1

By substituting ϕ for σ^2 in the proof of [2, Theorem 4.1], we see that $X(t)$ has LND(ϕ) if

$$\lim_{h\to 0} h^2/\phi(h)=0, \tag{3.1}$$

and, for every $m\geq 2$, and nonzero vector $(b_1,\ldots,b_m)$,

$$\liminf_{t_m\downarrow 0}\int_{-\infty}^{\infty}\left|\sum_{j=1}^m b_j\frac{e^{ixt_j}-e^{ixt_{j-1}}}{\phi^{1/2}(t_j-t_{j-1})}\right|^2\frac{1+x^2}{x^2}\,dF(x)>0. \tag{3.2}$$

Since (2.1) is identical with (3.1), it suffices to confirm (3.2). Following the proof in [2, p. 82], we will show that if there exists a sequence of m-tuples $(t_1, \ldots, t_m)$ with $0 = t_0 < t_1 < \cdots < t_m$ and $t_m \to 0$, such that

$$\lim \int_{-\infty}^{\infty} \left| \sum_{j=1}^{m} b_j \frac{e^{ixt_j} - e^{ixt_{j-1}}}{\phi^{1/2}(t_j - t_{j-1})} \right|^2 \frac{1+x^2}{x^2} \, dF(x) = 0, \tag{3.3}$$

then

$$b_1 = \ldots = b_m = 0. \tag{3.4}$$

From the sequence of m-tuples we can by compactness extract a subsequence such that the ratios $t_j - t_{j-1}/t_k - t_{k-1}$ all tend to nonnegative limits. Therefore, we may assume that this holds for the sequence satisfying (3.2). Put

$$h = \max_{1 \leq k \leq m} t_k - t_{k-1}. \tag{3.5}$$

In the case (1.4), Lemma 2.2 and the relation (3.3) imply

$$\lim \int_1^c \left| \sum_{j=1}^{m} b_j \frac{e^{ixt_j/h} - e^{ixt_{j-1}/h}}{[\phi(t_j - t_{j-1})/\phi(h/c)]^{1/2}} \right|^2 \frac{dx}{x} = 0, \tag{3.6}$$

for every $c > 1$. In the case (1.5), Lemma 2.3 and the relation (3.3) imply

$$\lim_{1/h \leq k \leq c/h} k^{-1} \left| \sum_{j=1}^{m} b_j \frac{e^{ikt_j} - e^{ikt_{j-1}}}{[\phi(t_j - t_{j-1})/\phi(h/c)]^{1/2}} \right|^2 = 0, \tag{3.7}$$

for every $c > 1$.

The proof of the theorem in the case (1.4) is now similar to the proof of [2, Theorem 6.1]. The only properties of ϕ that are needed to deduce (3.4) from (3.6) are those listed in Lemma 2.1. These correspond to the properties of the function $\sigma^2(t)$ used to pass from the relation [2], formula (6.5), to the conclusion at the bottom of page 83. These properties are stated as [2, Lemmas 5.2 and 5.3]. This completes the proof of Theorem 1.1 under (1.4).

We give a more detailed proof in the case (1.5). The following assertions are proved in exactly the same way as the corresponding statements in [2, page 82]:

(i) If

$$\lim (t_j - t_{j-1})/h = 0, \tag{3.8}$$

then the term of index j in the inner sum in (3.7) converges boundedly to 0.

By virtue of the relation

$$\sum_{1/h \leq k \leq c/h} 1/k \sim \log c, \tag{3.9}$$

statement (i) above implies that in calculating the limit in (3.7) we may eliminate all terms in the inner sum whose indices j satisfy (3.8), and consider only terms satisfying

$$\lim (t_j - t_{j-1})/h = \tau_j, \quad \text{where} \quad 0 < \tau_j \leq 1. \tag{3.10}$$

(ii) By passing to a subsequence, we may also suppose that

$$\lim \phi(t_j - t_{j-1})/\phi(h/c) = \phi_j, \quad \text{where} \quad 0 < \phi_j < \infty. \tag{3.11}$$

Statement (ii), together with (3.9), implies that we can replace the ratio $\phi(t_j - t_{j-1})/\phi(h/c)$ by its limit (3.11) in the relation (3.7). Hence, by replacing $b_j/\phi_j^{1/2}$ by b_j, we see that (3.7) is equivalent to

$$\lim \sum_{1 \leq kh \leq c} k^{-1} \left| \sum_{j=1}^{m} b_j(\mathrm{e}^{\mathrm{i}kt_j} - \mathrm{e}^{\mathrm{i}kt_{j-1}}) \right|^2 = 0. \tag{3.12}$$

Define

$$S_1 = \{j: 1 \leq j \leq m, \tau_j > 0\}. \tag{3.13}$$

We claim that the relation (3.12) is unchanged if the inner sum is restricted to $j \in S_1$. Indeed the term of index j is of modulus at most $b_j c(t_j - t_{j-1})/h$, so that when the squared modulus in (3.12) is expanded, the resulting contribution to the entire sum in (3.12) from the term of index j is of the order $(t_j - t_{j-1})/h$, which tends to 0 if $j \notin S_1$.

It follows from (3.10) that

$$\lim t_j/h = \tau_1 + \cdots + \tau_j. \tag{3.14}$$

By (3.9) the relation (3.12) is unchanged also if the exponents $\mathrm{i}kt_j$ are replaced by their asymptotic equivalents $\mathrm{i}kh(\tau_1 + \cdots + \tau_j), j \in S_1$. Therefore, (3.12) is equivalent to

$$\lim \sum_{1 \leq kh \leq c} k^{-1} \left| \sum_{j \in S_1} b_j(\mathrm{e}^{\mathrm{i}kh(\tau_1 + \cdots + \tau_j)} - \mathrm{e}^{\mathrm{i}kh(\tau_1 + \cdots + \tau_{j-1})}) \right|^2 = 0.$$

Define

$$F_h(x) = \sum_{1/h \leq k \leq x/h} k^{-1};$$

then the preceding limit relation is equivalent to

$$\lim \int_1^c \left| \sum_{j \in S_1} b_j(\mathrm{e}^{\mathrm{i}x(\tau_1 + \cdots + \tau_j)} - \mathrm{e}^{\mathrm{i}x(\tau_1 + \cdots + \tau_{j-1})}) \right|^2 \mathrm{d}F_h(x) = 0.$$

Since $F_h(x) \to \log x$ for $h \to 0$, the Helly-Bray lemma implies that the relation above is equivalent to

$$\int_1^c \left| \sum_{j \in S_1} b_j(\mathrm{e}^{\mathrm{i}x(\tau_1 + \cdots + \tau_j)} - \mathrm{e}^{\mathrm{i}x(\tau_1 + \cdots + \tau_{j-1})}) \right|^2 \frac{\mathrm{d}x}{x} = 0.$$

The argument following [2, (6.14)], permits the conclusion

$$b_j = 0, \quad j \in S_1. \tag{3.15}$$

Define

$$h = \max(t_j - t_{j-1} : 1 \leq j \leq m, j \notin S_1), \tag{3.16}$$

and, by analogy to (3.10)

$$\sigma_j = \lim(t_j - t_{j-1})/h \le 1, \quad j \notin S_1, \tag{3.17}$$

where, as before, the limit may be assumed to exist. Under the assumption (3.3) it has just been concluded that (3.15) holds; hence, the relation (3.3) is unchanged if the terms of indices $j \in S_1$ are excluded. Thus if h is defined by (3.16) instead of (3.5), the relation (3.3) implies the relation (3.12) without the terms of index $j \in S_1$:

$$\lim \sum_{k:\,1 \le kh \le c} k^{-1} \left| \sum_{j \notin S_1} b_j (\mathrm{e}^{\mathrm{i}kt_j} - \mathrm{e}^{\mathrm{i}kt_{j-1}}) \right|^2 = 0. \tag{3.18}$$

Define

$$S_2 = \{j\colon 1 \le j \le m, j \notin S_1, \sigma_j > 0\}; \tag{3.19}$$

then, by the same argument as that following (3.13), the relation (3.18) is unchanged if the index set in the inner sum is restricted to $j \in S_2$:

$$\lim \sum_{k:\,1 \le kh \le c} k^{-1} \left| \sum_{j \in S_2} b_j (\mathrm{e}^{\mathrm{i}kt_j} - \mathrm{e}^{\mathrm{i}kt_{j-1}}) \right|^2 = 0. \tag{3.20}$$

Our next goal is to show that (3.20) implies

$$b_j = 0, \quad j \in S_2. \tag{3.21}$$

The proof of this is more complicated than that of (3.15) because of the possible presence of pairs of indices j, $l \in S_2$ for which

$$(t_j - t_l)/h \to \infty. \tag{3.22}$$

In order to complete the proof of (3.21) by means of the reasoning in [2, p. 83], it suffices to show that the term of index (j, l) in the expansion of the square in (3.20) makes an asymptotically negligible contribution to the sum in (3.20) if (3.22) holds:

$$\lim \sum_{k:\,1 \le kh \le c} k^{-1} (\mathrm{e}^{\mathrm{i}kt_j} - \mathrm{e}^{\mathrm{i}kt_{j-1}})(\mathrm{e}^{\mathrm{i}kt_l} - \mathrm{e}^{-\mathrm{i}kt_{l-1}}) = 0. \tag{3.23}$$

For this purpose we prove

$$\lim_{n\to\infty, s\to 0, ns\to\infty} \sum_{n \le k < nc} k^{-1} \mathrm{e}^{\mathrm{i}ks} = 0. \tag{3.24}$$

(This takes the place of the Riemann–Lebesgue lemma in the absolutely continuous case considered in [2, p. 83].) Indeed, writing

$$k^{-1} = \int_0^\infty \mathrm{e}^{-kx}\,\mathrm{d}x,$$

we see that the sum in (3.24) is representable as

$$\int_0^\infty (1 - \mathrm{e}^{-x+\mathrm{i}s})^{-1} (\mathrm{e}^{-n(x-\mathrm{i}s)} - \mathrm{e}^{-[nc](x-\mathrm{i}s)})\,\mathrm{d}x,$$

which, by the substitution $y=x/s$, is equal to

$$\int_0^\infty s(1-e^{-s(y-i)})^{-1}(e^{-sn(y-i)}-e^{-s[nc](y-i)})\,dy.$$

By standard considerations this converges to 0 under the limiting operation defined in (3.24).

Under (3.17) and (3.22), it also follows that each of the ratios $(t_j-t_{l-1})/h$, $(t_{j-1}-t_l)/h$ and $(t_{j-1}-t_{l-1})/h$ tends to ∞. Thus the sum in (3.23), which is equal to a linear combination of four sums of the same form as in (3.24) with $n=1/h$, has, by (3.24), the limit 0. As noted after (3.22), this completes the proof of (3.21). The rest of the proof follows as in [2, p.83], and this completes the case (1.5).

Assuming the concept of local time in [2], we prove:

Theorem 3.1. *If, in addition to either* (1.7) *or* (1.8), *we assume*

$$\int_0^d \frac{dt}{(F(\infty)-F(1/t))^{1/2}}<\infty, \tag{3.25}$$

then the local time exists almost surely.

Proof. By the representation (1.3), and the relation (3.2) with $b_1=1$, $b_j=0$, $j=2,\ldots,m$, it follows that

$$\liminf_{t\to 0} \sigma^2(t)/(F(\infty)-F(1/t))>0.$$

Therefore, (3.25) implies

$$\int_0^d \frac{dt}{\sigma(t)}<\infty,$$

which, by [1], implies the existence of the local time. (While the statement in [1] is for stationary processes, it is also valid for processes with stationary increments.)

4. Application to random Fourier series

Let $X(t)$ and $Y(t)$ to be two processes, defined on possibly different probability spaces, such that $Y(t)$ is Gaussian with mean 0. Let $g(u)$, $u\geq 0$, be a nonnegative function. According to the definition in [3], we say that $X(t)$ is g-subordinate to $Y(t)$ if for every $m\geq 1$, time points $t_1,\ldots,t_m$ and real numbers $u_1,\ldots,u_m$,

$$\left|E\exp\left(i\sum_{j=1}^m u_jX(t_j)\right)\right|\leq g\left(\operatorname{Var}\sum_{j=1}^m u_jY(t_j)\right). \tag{4.1}$$

It follows in particular that

$$\left|E\exp(iu(X(t))-X(s))\right|\leq g(u^2\operatorname{Var}(Y(t)-Y(s))). \tag{4.2}$$

Lemma 4.1. *Let the Gaussian process $Y(t)$ have stationary increments, and suppose that its spectral function satisfies the conditions of Theorem* 3.1. *If*

$$\int_0^\infty g(u^2)\,\mathrm{d}u<\infty, \tag{4.3}$$

and X is g-subordinate to $Y(t)$, then $X(t)$ has a local time.

Proof. According to the criterion for the existence of the local time for a general process (see [1, pp. 283]) it suffices to show that

$$\int_0^d\int_0^d\int_{-\infty}^\infty |E\,\mathrm{e}^{\mathrm{i}u(X(t)-X(s))}|\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t<\infty.$$

By (4.2), it suffices to show that

$$\int_0^d\int_0^d\int_0^\infty g(u^2\,\mathrm{Var}(Y(t)-Y(s)))\,\mathrm{d}u\,\mathrm{d}s\,\mathrm{d}t<\infty.$$

The integral above is obviously equal to

$$2\int_0^\infty g(u^2)\,\mathrm{d}u\cdot\int_0^d (d-t)\frac{\mathrm{d}t}{\sigma(t)}\leqslant 2d\int_0^\infty g(u^2)\,\mathrm{d}u\int_0^d\frac{\mathrm{d}t}{\sigma(t)}.$$

According to (4.3) and the proof of Theorem 3.1, the latter is finite.

Let $X_n, n\geqslant 0$, and Y_n, $n\geqslant 0$, be independent standard normal random variables, and a_n, $n\geqslant 0$, real numbers such that

$$\sum_n a_n^2<\infty. \tag{4.4}$$

Then, for each t, the series

$$Y(t)=\sum_n a_n(X_n\cos nt+Y_n\sin nt) \tag{4.5}$$

converges with probability 1, and $Y(t)$, $-\infty<t<\infty$, represents a stationary Gaussian process with mean 0 and covariance function

$$EY(s)Y(s+t)=\sum_n a_n^2\cos nt=r(t), \tag{4.6}$$

and with the spectral function F given by (1.4) with $f_n=a_n^2$.

Now we define a process $X(t)$ of the same form as $Y(t)$ in (4.5) except that X_n and Y_n are not necessarily Gaussian.

Theorem 4.1. *X_n, $n\geqslant 0$, and Y_n, $n\geqslant 0$, be independent random variables with a common distribution function having the characteristic function $\psi(u)=E(\exp(\mathrm{i}uX_0))$. If*

$$\sum_n\sup_{|u|\leqslant 1}|1-\psi(ua_n)|<\infty, \tag{4.7}$$

then the series

$$X(t)=\sum_n a_n(X_n \cos nt + Y_n \sin nt) \tag{4.8}$$

converges almost surely for each t. If

$$0<|\psi(u)|<1 \quad \text{for all } u\neq 0, \tag{4.9}$$

and, for some α, $0<\alpha\leq 2$,

$$\liminf_{|u|\to\infty \text{ or } u\to 0} |u|^{-\alpha}|\log\psi(u)|>0. \tag{4.10}$$

then the process $X(t)$ *is g-subordinate to the corresponding Gaussian process* (4.5) *with*

$$g(u)=\exp(-c|u|^{\alpha/2}) \tag{4.11}$$

for some $c>0$.

Proof. Consider the first portion of the series (4.8)

$$\sum_n a_n X_n \cos nt.$$

Since the characteristic function of the typical term is $\psi(a_n u \cos nt)$, a classical criterion implies that the series converges almost surely if

$$\operatorname{mes}\{u : \sum_n |1-\psi(a_n u \cos nt)|<\infty\}>0.$$

The latter is implied by the assumption (4.7). The proof for the terms $a_n Y_n \sin nt$ is the same, and so (4.8) converges, as claimed.

The assumptions (4.9) and (4.10) imply

$$|\psi(u)|\leq \exp(-c|u|^{\alpha}), \tag{4.12}$$

for some $c>0$. The subordination of $X(t)$ to $Y(t)$ is now a direct consequence of [3, Theorem 6.1], with g as in (4.11).

Example. Let X_n and Y_n have symmetric stable distributions of index α, $0<\alpha<2$; here $\psi(u)=\exp(-b|u|^{\alpha})$, for some $b>0$. Let the sequence (a_n) satisfy

$$\sum_n a_n^{2\alpha}<\infty. \tag{4.15}$$

Then conditions (4.7), (4.9) and (4.10) are satisfied.

The spectral function of the Gaussian process (4.5) satisfies the conditions of Theorem 3.1 if $f_n=a_n^2$ satisfies (1.8) and if

$$\int_0^d \frac{\mathrm{d}t}{\{\sum_{n>1/t} a_n^2\}^{1/2}}<\infty. \tag{4.16}$$

We conclude from Lemma 4.1 and Theorem 4.1, and from the main result of [3] that the process $X(t)$ has a local time and that the properties of the local time of the Gaussian counterpart $Y(t)$ obtained from $\mathrm{LND}(\phi)$ are inherited by the local time of $X(t)$. In the particular case of a sequence (a_n) such that $a_n\sim n^{-\beta/2}$, the conclusions above hold for

$$\max(1,\alpha^{-1})<\beta<3.$$

References

[1] S.M. Berman, Local times and sample function properties of stationary Gaussian processes, Trans. Amer. Math. Soc. 137 (1969) 277-299.

[2] S.M. Berman, Local nondeterminism and local times of Gaussian processes, Indiana U. Math. J. 23 (1973) 69-94.

[3] S.M. Berman, Local times for stochastic processes which are subordinate to Gaussian processes, J. Multiv. Anal. 12 (1982) 317-334.

[4] J. Cuzick, Local nondeterminism and the zeros of Gaussian processes, Ann. Probab. 6 (1978) 72-84; 15 (1987) 1229 (correction).

[5] L. de Haan, On Regular Variation and its Application to the Weak Convergence of Sample Extremes (Mathematisch Centrum, Amsterdam, 1970).

[6] R.N. Miroshin, Local regularity criteria for stochastic processes, Vestnik Leningrad Univ. (1983) 69-74.

[7] E. Seneta, Regularly Varying Functions, Lecture Notes in Mathematics 508 (Springer, Berlin, 1976).